REPRESENTATIONS AND STRATEGIES IN LINEAR EQUATION SOLUTIONS TAUGHT WITH CONCRETE MATERIALS

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ABSTRACT

A class of grade 9 students underwent a period of instruction on the use of concrete materials to represent algebraic expressions and equations and to solve linear equations. Interviews conducted at the end of the instruction aimed at studying the students' knowledge of variables and their ability to solve equations using a variety of methods. In particular cognitive load theories were used to explain the problems in teaching and learning of algebra using concrete materials. The hypothesis that the students benefit from concrete material in algebra is not supported by their use in this study.

There has been a great deal of contemporary cognitive research in early mathematics learning, particularly in number and arithmetic. Case and Sowder (1990) and Case and Griffin (1990) have used Case's theory to explain developmental sequences on the basis of the complexity of the concepts in interaction with children's increasing capacity to process information. Boulton-Lewis (1987, 1993a, 1993b, in press b), Boulton-Lewis and Halford (1992), Boulton-Lewis and Tait (in press) and Halford and Boulton-Lewis (1992) have applied Halford's structure mapping theory of cognitive development to assessing the effect of the processing load of representations and strategies in selected aspects of early mathematics learning, such as measurement, place value, addition and subtraction. This research has explained some learning difficulties and use of alternative child-chosen strategies in terms of the cognitive overload imposed by contemporary teaching strategies and material usage.

In comparison, cognitive research into secondary school mathematics topics, such as algebra, has been limited and narrowly defined (Chaiklin, 1989). Biggs and Collis (1982) have discussed algebraic concepts and classified student responses to algebraic equations in relation to the SOLO (Structure of the Observed Learning Outcome) Taxonomy categories: unistructural, multistructural, relational or extended abstract levels (cf. Biggs, 1991). Their conclusion, based on Collis's (1975) research, was that understanding pronumerals 'seemed to depend on what they [students] were able to regard as real for them': unistructural responses mapped the pronumeral directly into a specific number, multistructural mapped pronumerals into a few sets of numbers and relational conceptualised pronumeral as generalised number, while extended abstract perceived the pronumeral as an abstract variable (and algebra as an 'abstract self-consistent system').

Halford (1987, 1993) placed elementary algebra, with its combinations of binary operations, eg., a(b+c)=d, at the multiple system mapping level of his structure mapping theory. Halford and Boulton-Lewis (1992) argued that the recognition of correspondences between the manipulation of operation symbols and parentheses in arithmetic expressions and algebraic rules depended on a series of multiple system mappings with concomitant high processing loads. To reduce load, Halford (1993) posited that it is necessary to learn each correspondence before progressing to the next, however, this prediction has not been tested empirically.

Cognitive load has been considered in relation to instruction. Cooper and Sweller (1987), Sweller and Cooper (1985) and Sweller and Low (1992) have argued, for example, that the cognitive load imposed by means-ends problem-solving strategy interferes with novices' learning of algebraic procedures. They have contended that worked examples reduces the cognitive load and is more effective. Boulton-Lewis and Halford (1992) have argued that early mathematics difficulties might be due to the load imposed by lack of familiarity with symbols, procedures and representations. Furthermore, they have argued that inappropriate teacher choice of representations and procedures might impose a load that makes tasks difficult, meaningless or actually mathematically incorrect.

Current curriculum approaches, eg. Quinlan, Low, Sawyer and White (1993), are suggesting that teachers use concrete and other representations to introduce concepts such as variables in algebra. The implication of cognitive research is that such approaches may be ineffective unless cognitive load is catered for. To this end, a project has been undertaken to study the interaction of prior knowledge, use of representations, solution strategies and cognitive load in solving linear equations. This paper reports on the preliminary results from the pilot study in the project which investigated the effect of instruction with concrete and pictorial representations on students' ability to represent linear expressions and equation with concrete material and their solution of linear equation using these representations. Further, this analysis will identify some of he common problems/misconceptions displayed by students while making these representations.

METHOD

Subjects

Twenty-one students from a Year 9 class of mixed ability students in a middle-class suburban state secondary school with a reputation for high academic standards and innovative mathematics teaching. Previously the class has completed an algebra unit on variables as generalisations through patterning. The class was observed for this study while learning the concept of a variable as unknown quantity using the concrete representation of cups and counters; and linear equations and their solution using these representations.

Instruments

The instrument was an interview given to the students before and after lessons on linear algebra were observed. Each interviews involved six tasks: (a) pattern and generalisation; (b) meaning of variable; (c) differences between expression and equation; (d) meaning of equals; (e) solution of linear equations; (f) inverse operations; and (g) questions requiring the use of concrete and pictorial representations. For this paper, only data from the post treatment interview and only sections (b) and (e) are pertinent. For (b), the students were asked what letters in expressions were called and what they stood for. For (e), students were asked to solve two linear equations $2x+5=17$ and $3x-4=-13$. If students did not voluntarily use representations to solve the linear equations, they were asked to solve again with concrete and then pictorial representations.

In addition to the interviews, classroom intreactions were videotaped and transcribed. This unit of work was covered in four lessons. Classroom observations will be analysed at a later date to investigate possible relationships between classroom teaching and student performace on the interviews tasks.

Procedure

Students in this study took part in a pre-treatment interviews on algebra tasks described above. Each interview was videotaped and took about 30 minutes. The treatment consisted of four sessions on the representation by concrete material of linear equations and their solution. One month after the completion of these lessons, twenty-one students were video-taped completing the above interview.

The lessons developed a model consisting of green discs to represents units, yellow discs to represent negative units, white cups to represent variables and yellow cups to represent the negative value of variables. There was some inconsistency on the part of the teacher in using either the cup or the unknown number of discs inside it to represent the variable. The lessons then introduced the balance image using cups and discs to represent equations. The notions of upsetting and restoring balance and sharing counters equally between cups were used to solve equations. All through the lessons the material was available for students at the front of the room if and when they required it. Very few of them asked to use the materials in solving classroom examples prefering the mental and or the diagram procedures.

RESULTS

Interview protocols were analysed to determine (a) categories of knowledge with respect to variables, equal sign, operations and equations; (b) categories of strategies used to solve linear equations; and (c) achievement in solution of linear equations. In the following analysis, only part of the data from the second set of interviews will be addressed. More detailed analysis of the results from the study are forthcoming.

The meaning and representation of variables

All the students were able to say that letters in expressions stood for unknown numbers. However, only one of the 21 students actually used the word variable to name these letters and six of the students indicated that the letters stood for objects as well as unknown numbers. Furthermore, only 4 students could accurately represent linear expressions with cups and counters as they had been taught. About half the students represented the letters with counters, different in colour from the counters representing the numbers. The remaining 7 students could not do any representation using concrete materials.

Solving equations with materials

Fourteen students answered the first linear equation, $2x+5=17$, correctly and 9 the second, $3x-4=-13$. No students voluntarily used material in their attempts to solve the first equation and only 2 for the second. Eight students voluntarily used the balance strategy, which was taught in the third lesson with the concrete and pictorial materials. Thirteen students voluntarily used the inverse strategy, which was raised at the end of the fourth lesson, for the first equation and 9 for the second. It became clear from the classroom interactions that students have studied this method to solve equations in the previous year.

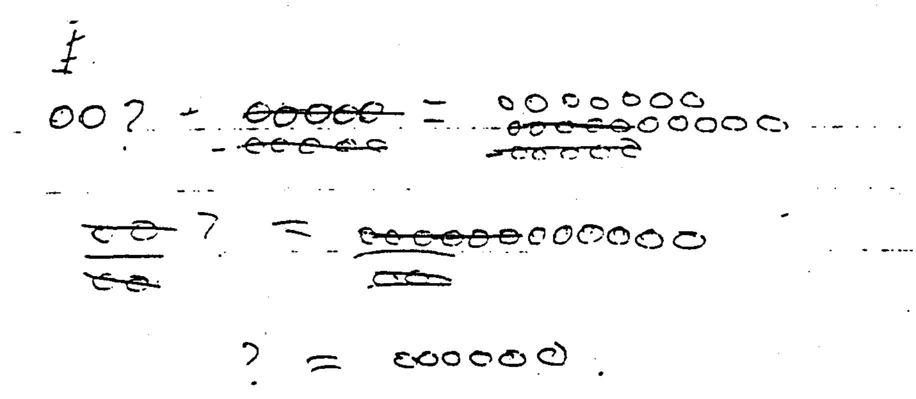
When asked, only one student could use representations to generate a solution for the first equation, ie. use the cups and counters and balance strategy to determine, without using previously gained knowledge of the answer. Of the remaining 20 students, 7 were unable to use materials and 13 students used the materials to illustrate the solution they had previously obtained by other methods. The illustrating was done in relation to the previous strategy used, eg., by starting with two groups of 6 counters (in the cups for some students) and adding 5 to get 17 for the balance strategy, and by starting with 17 counters, subtracting by 5 and dividing by 2 (without using any cups) for the inverse strategy. For the second equation, only 9 students felt able even to attempt material usage, however, 5 used the material to generate the answer.

Students were also asked to solve the same equation using the balance charts developed in the third lesson in class. Only 9 students could get the answer for the first equation and 6 for the second equation. Hence, fewer students could generate the answer using balance charts than mental algorithms.

Errors in modelling with materials

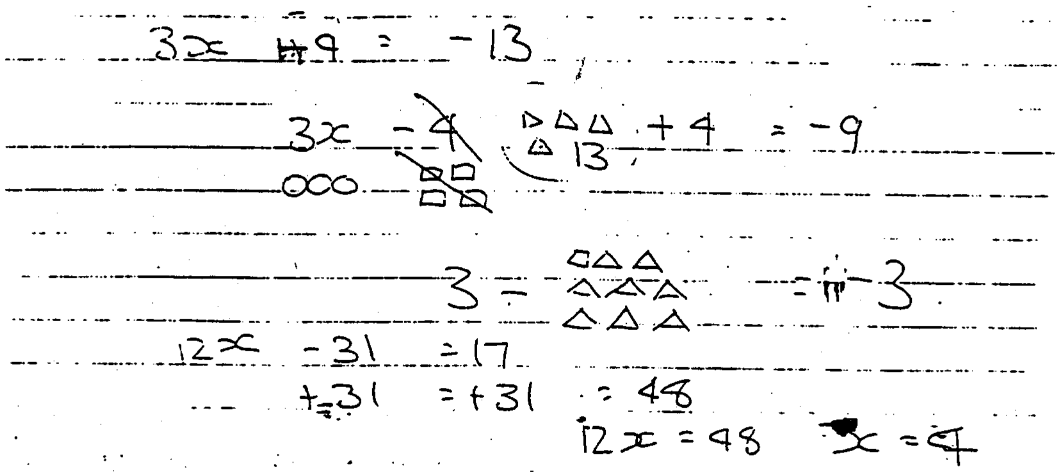
We summarise here the most common mistakes and misconceptions that students exhibited in modelling the expressions and equations with materials and in pictorial diagrams.

1. *Literal modelling:* A very common source of error in representing variables with concrete materials is to translate the expression literally term by term. For example, the expression 5a was represented by five discs followed by a cup, rather than five cups, or representing the expression 2x+5 by 2 discs (for two 2x) a yellow disc for the + and 5 green discs for the 5. Anita's solution for the equation 2x+5=17 illustrates the literal representation by material as well as the use of materials by some children to illustrate the solution obtained by other methods rather than to generate the solution.



2. *Variables not modelled*: When asked to model the expression 5a some students placed five discs on the table. For these students, a disc did not stand for the

 variable but for a unit. When asked about the representation of the variable "a" these students said that it is not possible to represent a value that is not known. A related mistake is to represent the variable and the constants by the same objects. For example, one student represented the expression 2x+5 by two groups of 2 and 5 discs separated by a short distance. Liz representation of the equation $3x-4={}^{-}13$ illustrates this problem:



3. *Two sides of the balance*: In modelling the equation $3x-4={}^{-}13$, one students represented the 3x by three cups on the left hand side of the balance and the -4 by 4 yellow disc on the right hand side of the balance. the $^{-}13$ was not modelled.

DISCUSSION AND CONCLUSIONS

Observations

Two difficulties were evident in the instructional use of representations: there was inconsistency in how variables were represented: either as a cup or as the unknown quantity inside the cup. For example, during the first lesson, the teacher was using the physical representations to show that 2x+3 does not equal 5x by asserting that we cannot add cups and discs. Secondly, there was a lack of connection drawn between the modelling and the mental algorithm. The failure to relate sharing counters between cups to division in solving the equation 2x=6 is an example of this difficulty. This lack of connection was exacerbated by the teacher introducing symbolic methods as "short cut" and in providing a final short cut, at the end of the fourth lesson, which was based the inverse operations, a different solution strategy from that being used up to that time with the materials. No attempt was made to connect the two strategies to show that they are different.

Physical models exemplify mathematical concepts and can be used to develop mental models. Teachers need to be aware of the strengths and weaknesses of each concrete model that they use. Not drawing connections between different representations and ignoring the complexity in materials, such as different coloured cups and counters representing negative values, leads to difficult situations where teachers have to revert to rules for materials, eg., changing the colours of discs and cups when multiplying by a negative number outside brackets and this all increases the processing load.

Interviews

In general, after instruction on the representation of expressions using concrete materials, students could not use the same material to represent expressions and variables correctly in interview situations. did not voluntarily use either the materials approach or the solution strategy ("balancing") taught in the lessons. When directed to do so, they were less successful than with their chosen method. They were also much slower and more hesitant in the material usage than with symbols. They tended to use materials and pictures to illustrate their answers obtained from symbolic algorithms. Evidence that these students have benefited from instruction on material representations is not very strong.

Conclusions

Overall, students had poor knowledge on which to build their equation solutions. This combined with the complexity of the material and the pictures used would, from the theories described before, place significant cognitive load on the students. One method of reducing this load is to adopt strategies that simplify the process. The "short cut" of the fourth lesson, which was really an alternative approach to solving equations, does this and was adopted by most students. In the interviews, when directed to use the techniques of the previous three lessons, students exhibited behaviour consistent with that expected under cognitive stress, eg. lower achievement, slower and more hesitant activity, and attempts to recall the steps in a previously memorised procedure rather than generating the solution with understanding. Arguably, the use of the material may have added to the cognitive demand on students because it involved complex set of rules that need to be recalled when representing equations and there was no suystematic attempts to develop connections between the material models and the mental models.

The result of this was that students ignored the instruction in the lessons and adopted a solution technique which gave answers but did not necessarily exhibit understanding as envisaged by the teacher. Hiebert and Carpenter (1992) described such behaviour in terms of poorly understood concepts and a lack of connections between knowledge areas and representations.

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